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Abstract. Generalized symmetries with arbitary functions of time t for the well known 2+1-dimensional integrable model, Kadomtsev–Petviashvili (KP) equation, are found by means of the extended mastersymmetry approach. Then an explicit and simple constructive formula for the symmetries of the KP equation is derived directly from the symmetry definition equation, without using complicated recursion operators. All the known symmetries appear as special cases of those obtained in this paper. The general infinite-dimensional Lie algebra constituted by these symmetries is also given.

1. Introduction

In the past two decades, the discovery of soliton solutions for certain 1+1-dimensional nonlinear evolution equations with physical applications has aroused great interest and attention among physists and mathematicians [1]. One of the important developments is that nonlinear models such as the KdV and Liouville equations, in general possess many sets of infinitely many symmetries [2–6]. There exist different types of effective methods to obtain the symmetries of an evolution equation. One of them uses Lie algebras of mastersymmetries to obtain all commuting symmetries.

Fuchssteiner used mastersymmetries to get both the time-independent and time-dependent symmetries of the Kadomtsev–Petviashvili (KP) equation [7]. In this paper, we modify the mastersymmetry method of [7] to give the more general symmetries of the KP equation and the infinite dimensional Lie algebra constructed by the generalized symmetries. Section 3 is devoted to presenting an explicit expression for all the symmetries obtained in section 2. Section 4 is a summary and discussion.

2. Symmetries and Lie algebra of the KP equation

Fuchssteiner [7] pointed out that

$$G_3 \equiv y^2 \tag{1}$$

is a mastersymmetry of degree three for the KP equation

$$u_{xt} = (6uu_x - u_{xxx})_x - 3u_{yy} \equiv K_{2x} \tag{2}$$

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because

$$[[[G_3, K_2], K_2], K_2] = [[G_2, K_2], K_2] = [G_1, K_2] = 108\, K_2 \tag{3}$$

where the commutator $[A, B]$ is defined by

$$[A, B] = \frac{\partial}{\partial \varepsilon}[A(u+\varepsilon B(u)) - B(u+\varepsilon A(u))]|_{\varepsilon=0} = A'B - B'A \tag{4}$$

and

$$K_2 = D_x^{-1}K_{2x} = 6uu_x - u_{xxx} - 3D_x^{-1}u_{yy} \tag{5}$$

with

$$D_x^{-1} = \int_{-\infty}^{x} \mathrm{d}x \tag{6}$$

being the inverse of the differential operator $D_x = \partial/\partial x$. Applying to a constant C, he defined the Lie algebraic meaning of D_x^{-1} by

$$D_x^{-1}C = xC. \tag{7}$$

However, in order to get the most general results, we should define the D_x^{-1} more generally as

$$D_x^{-1}C = xC + f(y, t), \tag{8}$$

where f is an arbitary function of y and t.

Now we begin the calculation with the known seed symmetry K_2 and the mastersymmetry $G_3 = y^2$ as in [7], but with the definition (7) for D_x^{-1} replaced by (8). Explicit computation yields

$$\begin{aligned} G_2 &= [G_3, K_2] = -6u_x y^2 + 6x + f(y, t), \\ G_1 &= [G_2, K_2] = -36(xu_x + 2u + 2yu_y) - 6fu_x + 3f_{yy}x + f_1(y, t). \end{aligned} \tag{9}$$

Finally, we have

$$\begin{aligned} [G_1, K_2] &= 108K_2 - 6f_1u_x + f_2(y, t) - 18(f_{yy}xu_x + 2f_yu_y + 2f_{yy}u) + (9/2)f_{yyyy}x^2 + 3f_{1yy}x \\ &\equiv 108K_2 + \sigma_0 \end{aligned} \tag{10}$$

where f, f_1 and f_2 are three integral functions which should be fixed from the symmetry definition equation of the KP equation such that y^2 is still a mastersymmetry of degree 3 when D_x^{-1} is given by (8). The symmetry definition equation of the KP reads

$$\sigma_{xt} = [(6D_x^2u - D_x^4) - D_y^2]\sigma = K'_{2x}\sigma \tag{11}$$

or equivalently

$$\sigma_t = D_t\sigma = K'_2\sigma = (6D_xu - D_x^3 - D_x^{-1}D_y^2)\sigma \tag{12}$$

where D_t denotes the total t-derivative.

Substituting σ_0 given in (10) into (11), we get a first generalized seed symmetry

$$K_0(h) = h(t)u_x + (1/6)\dot{h}(t) \tag{13}$$

with $f_{1y} = f = f_{2y} = 0$, $h = -6f_1(t)$, $f_2 = (1/6)\dot{h} = D_th/6$ and h being an arbitrary function of t. When $h=1$, $K_0(h)$ reduces to the known $K_0(1)=K_0=u_x$, corresponding to the x-translation. Now using the known symmetry [7]

$$K_3=4[4uu_y+2u_xD_x^{-1}u_y-D_x^{-2}u_{yyy}-u_{xyy}]$$

and the generalized seed (13), we obtain

$$K_1(h)=(3/4)[K_0(f),K_3]=-2hu_y+(1/3)\dot{h}yu_x+(1/18)\ddot{h}y. \tag{14}$$

where $h=\dot{f}$ and an integral function has been fixed as $(1/18)\ddot{h}y$ such that $K_1(h)$ satisfies (11). $K_1(h)$ is the generalization of the known y-translation invariance $K_1(1)=K_1=-2u_y$. Similarly

$$\begin{aligned}K_2(h)&=(3/4)[K_1(f),K_3]\\&=-hu_t-(2/3)\dot{h}yu_y-(1/3)\dot{h}xu_x+(1/18)\ddot{h}y^2u_x-(2/3)\dot{h}u-(1/18)\ddot{h}x\\&\quad+(1/108)\dddot{h}y^2,\ (h=\dot{f})\end{aligned} \tag{15}$$

and

$$\begin{aligned}K_3(h)&=(3/4)[K_2(f),K_3]=hK_3+(1/3)\dot{h}(yu_{xxx}+2xu_y+4D_x^{-1}u_y+3yD_x^{-1}u_{yy}-6yu_x)\\&\quad-(1/9)\ddot{h}(xyu_x+2yu+y^2u_y)-(1/162)\dddot{h}(3xy-y^3u_y)\\&\quad+(1/972)\ddddot{h}y^3(h=\dot{f}).\end{aligned} \tag{16}$$

$K_2(h)$ is the generalization of the t-translation invariance, $K_2(1)=K_2=-u_t$, while $K_3(h)$ is the generalization of the known time independent symmetry $K_3(1)=K_3$.

Finally, by means of the known time-independent symmetries K_n [7] and generalized symmetry $K_2(h)$ or $K_3(h)$, we obtain all the generalizations of K_n:

$$K_n(h)=(3/(n+1))|K_2(f),K_n| \tag{17}$$

$$=(3/n)[K_3(f),K_{n-3}] \qquad (\dot{f}=h). \tag{18}$$

It is interesting that when we take $h=t$ for $K_3(h)$, then the generalized symmetry $K_3(h)$ reduces to the known time-dependent (linear in t) symmetry

$$K_3(t)=tK_3+(1/3)(yu_{xxx}+2xu_y+4D_x^{-1}u_y+3yD_x^{-1}u_{yy}-6yu_x)\equiv tK_3+\tau_{1,3} \tag{19}$$

where $\tau_{1,3}$ is just the mastersymmetry given in ref. [7] (up to a constant factor). Then (18) reduces to the known results:

$$K_n(1)=K_n=(3/n)[K_3(t),K_{n-1}]=(3/n)[\tau_{1,3},K_{n-1}]. \tag{20}$$

After finishing the detailed calculations we can see that the generalized symmetries obtained here constitute also a closed infinite dimensional Lie algebra

$$[K_n(h_1),K_m(h_2)]=(1/3)K_{n+m-2}((m+1)\dot{h}_1h_2-(n+1)\dot{h}_2h_1). \tag{21}$$

Here we would like to present some special cases of (21) to demonstrate its correctness, instead of giving tedious concrete verification.

(1) $h=1$. In this special case, all the $K_n(h)$ reduce to the known symmetries [8] $K_n=K_n(1)$ which can be derived from the conserved quantities [9] that do not depend explicitly on the variables x, y and t. All of these quantities commute.

(2) $h=t$. In this case, all the symmetries $K_n(t)\equiv\tau_n$ are just the so-called 'new symmetries' or 'τ symmetries' that depend explicitly on the variables x, y and t. All of these types of symmetries constitute a Virasoro algebra

$$[K_n(t),K_m(t)]=[\tau_n,\tau_m]=(1/3)(m-n)\tau_{n+m-2},n,m>0). \tag{22}$$

(3) $h(t)=t^m$, $(m\geqslant 1)$. This type of solutions is also known in the literatures [7, 11]. The algebra (21) is reduced to

$$[K_n(t^r), K_m(t^s)]=(1/3)[r(m+1)-s(n+1)]K_{n+m-2}(t^{r+s-1}) \qquad (n, m\geqslant 0, r, s\geqslant 1) \quad (23)$$

which was also given in [11]. In fact, it is quite natural to extend (23) to (21). If we restrict $h(t)$ to any analytical function then writing $h(t)$ in Taylor series form, one can prove (21) from (23) because (11) is linear.

(4) Let

$$K_0(h)\equiv Z(h)=hu_x+(1/6)\dot{h} \quad (24)$$

$$K_1(g)\equiv Y(g)=-2gu_y+(1/3)\dot{g}yu+(1/18)\ddot{g}y \quad (25)$$

and

$$K_2(f)\equiv X(f)=-fu_t-(2/3)\dot{f}yu_y-(1/3)\dot{f}xu_x+(1/18)\ddot{f}y^2u_x-(2/3)\dot{f}u-(1/18)\ddot{f}x + (1/108)\dddot{f}y^2 \quad (26)$$

where h, g and f are three arbitary functions of t. Then we get an infinite-dimensional subalgebra of the KP equation from eq. (21):

$$[Z(h_1), Z(h_2)]=0 \quad (27)$$

$$[Z(h), Y(g)]=0 \quad (28)$$

$$[Y(g_1), Y(g_2)]=(2/3)Z(\dot{g}_1g_2-\dot{g}_2g_1) \quad (29)$$

$$[Y(g, X(f)]=(1/3)Y(3\dot{g}f-2\dot{f}g) \quad (30)$$

$$[Z(h), X(f)]=(1/3)Z(3\dot{h}f-\dot{f}h) \quad (31)$$

$$[X(f_1), X(f_2)]=X(\dot{f}_1f_2-\dot{f}_2f_1). \quad (32)$$

Using a REDUCE package, this subalgebra was first given by Schwarz [12] where $X(f)$, $Y(g)$ and $Z(h)$ were expressed as the following totally equivalent forms

$$Z(h)=h(t)\frac{\partial}{\partial x}-(1/6)\dot{h}\frac{\partial}{\partial u} \quad (33)$$

$$Y(g)=2g(t)\frac{\partial}{\partial y}+(1/3)\dot{g}y\frac{\partial}{\partial x}-(1/18)\ddot{g}y\frac{\partial}{\partial u} \quad (34)$$

$$X(f)=-f(t)\frac{\partial}{\partial t}-(2/3)\dot{f}y\frac{\partial}{\partial y}-[(1/3)\dot{f}x-(1/18)\ddot{f}y^2]\frac{\partial}{\partial x} + [(2/3)\dot{f}u+(1/18)\ddot{f}x-(1/108)\dddot{f}y^2]\frac{\partial}{\partial u} \quad (35)$$

and the corresponding Lie product is charged as

$$[A, B]=AB-BA. \quad (36)$$

More details of this algebra have been studied in [13]. Three general types of similarity solutions of the KP equation were obtained using this subalgebra [13].

Because the integral functions which should be fixed have been included in every high–order symmetry, to obtain the concrete form of $K_n(h)$ from (17) or (18) is still quite difficult. This is especially true for the integration functions, though they coincide with that obtained by [11] because they have to be determined directly from the definition equation (11). We would like to give a simple and explicit formula to obtain $K_n(h)$.

3. A constructable formula for the symmetries of the KP equation

From equations (17) (or (18)) and (11) (or (12)), we know that the h (and its derivatives) dependence for $K_n(h)$ must be linear because (11) is linear in σ, equation (17) is linear in $h(=\dot f)$ and h is an arbitrary function of t. Accordingly, we know that $K_n(h)$ can only have the form

$$K_n(h)=\sum_{k=0}^{n+1} h^{(n+1-k)}K_n[k] \tag{37}$$

where $h^{(k)}=(D_t^k h(t))$, $K_n[k]\,(k=0,1,\ldots n+1)$ are functions of x, y, u and its derivatives, but are not time-dependent explicitly. The explicit time dependence of $K_n(h)$ has been separated out in $h^{(n+1-k)}$.

Substituting (37) into the left side of the definition equation (11) directly, we have

$$\begin{aligned} K_{nxt}(k) &= \sum_{k=0}^{n+1} k^{(n+2-k)}K_{nx}[k]+\sum_{k=0}^{n+1} h^{(n+1-k)}K_{nxt}[k] \\ &= \sum_{k=0}^{n+1} h^{(n+2-k)}K_{nx}[k]+\sum_{k=1}^{n+2} h^{(n+2-k)}K_{nxt}[k-1] \end{aligned} \tag{38}$$

while the right-hand side of (11) now becomes

$$K'_{2x}K_n(h)=K'_{2x}\sum_{k=0}^{n+1} h^{(n+1-k)}K_n[k]=\sum_{k=1}^{n+2} h^{(n+2-k)}K'_{2x}K_n[k-1]. \tag{39}$$

Because h is an arbitrary function of t, comparing the coefficients of $h^{(n+2-k)}$ of (38) and (39) we have

$$K_{nx}[0]=0 \tag{40}$$

for $k=0$, which means

$$K_n[0]=g_n(y) \tag{41}$$

is not only t-independent but also x- (and then u-) independent.

For $k=1,2,\ldots,n+1$, we have

$$K_{nx}[k]+K_{nxt}[k-1]=K'_{2x}K_n[k-1] \tag{42}$$

$$K_{nx}[k]=(K'_{2x}-D_xD_t)K_n[k-1]. \tag{43}$$

Solving (43) recursively yields

$$\begin{aligned} K_n[k] &= D_x^{-1}(K'_{2x}-D_xD_t)K_n[k-1] \\ &= (K'_2-D_t)K_n[k-1] \\ &= (K'_2-D_t)^2K_n[k-2] \\ &\quad \ldots \\ &= (K'_2-D_t)^kK_n[0] \\ &= (K'_2-D_t)^kg_n(y). \end{aligned} \tag{44}$$

For $k=n+2$, the condition reads

$$K_{nxt}[n+1]=K'_{2x}K_n[n+1] \tag{45}$$

which means $K_n[n+1]$ itself should be a time-independent symmetry.

Now the only thing left to do is to substitute

$$K_n[n+1]=(K'_{2r}-D_t)^{n+1}g_n(y) \tag{46}$$

into (45) to determine the only unknown function $g_n(y)$. However, because we know that the solution for this problem exists from the discussions of (23), and also that the solution can only possess the form (37), here we would like to fix $g_n(y)$ in a simple way.

If we say that x has a dimension $[x]$, then from the KP equation (2) we know that y, t and u should have the dimensions $[x]^2$, $[x]^3$ and $[x]^{-2}$ respectively. From equations (13)–(18), one can see that $K_n(h)/h$ must have dimension $[x]^{-n-3}$. That is to say, $h^{(n+1)}g_n(y)/h$ must have dimension $[x]^{-n-3}$. Accordingly, the only possible form for $g_n(y)$ is

$$g_n(y)=A_n y^n, \tag{47}$$

where A_n is a constant which is not very important because the symmetry equation (11) is linear.

Finally, we obtain the constructive formula for $K_n(h)$:

$$K_n(h)=(1/(2n!3^{n+1}))\sum_{k=0}^{n+1} h^{(n+1-k)}(K'_2-D_t)^k y^n \tag{48}$$

where the constant A_n has been fixed as $(1/(2n!3^{n+1}))$ so that $K_n(h)$ given by (48) and that determined by (18) are the same.

Starting from the general symmetry expression (48), we can obtain not only the explicit expressions of the commuting symmetries

$$K_n=K_n[n+1]=(1/(2n!3^{n+1}))(K'_2-D_t)^{n+1}y^n \tag{49}$$

but also the explicit expressions of the time-independent mastersymmetries of degree k:

$$\tau_{k,n}=(1/(2n!3^{n+1}))(K'_2-D_t)^{n+1-k}y^n \qquad (k=1,2,\ldots,n+1). \tag{50}$$

By means of (50), the general time-dependent symmetries $K_n(h)$ for the KP equation can be written as

$$K_n(h)=\sum_{k=0}^{n+1} h^{(k)}\tau_{k,n} \tag{51}$$

with $\tau_{0,n}=K_n$.

In concrete calculations we find that if we take $K_n(h)$ to have the form (48), the definition equation (7) for D_x^{-1} can be used. Actually, the integral functions in (17) or (18) have been fixed because we have fixed $K_n[0]$ for all $n\geqslant 0$.

4. Summary and discussions

In this paper, after extending the mastersymmetry method [7] for the KP equation, we have obtained an infinite set of time-dependent symmetries in which an arbitrary function of t has been included for every high-order symmetry. All the symmetries known can be considered as special cases. Though the time-independent symmetries can be obtained recursively in two different way by means of two recursion operators

[14], the expressions for these commuting symmetries are still quite complicated. We have given an explicit simple expression for the generalized time-dependent symmetries with an arbitrary function of t. The time-independent symmetries and master-symmetries of degree $k(k=0, 1, 2, \ldots, n+1)$ can all be obtained from this expression.

Together, these infinitely many symmetries constitute an infinite-dimensional Lie algebra. The infinite-dimensional Lie algebra with three arbitrary functions of t obtained by Schwartz [12] and discussed by David *et al* [13] is only a subalgebra of that given here. The algebra constituted by infinitely many generalized symmetries includes infinitely many arbitrary functions of t because all the functions in high-order symmetries can be independent of each other. Using the subalgebra with three arbitrary functions of t, David *et al* [13] obtained three types of solutions of the KP equation with three arbitrary functions. Actually, it is known that there exist some types of solutions with more than three arbitrary functions of t for the KP equation [15, 16]. Furthermore, the fact that infinitely many arbitrary functions of t exist for the symmetry group of the KP equation means that there may be some types of solutions with infinitely many arbitrary functions of t. How to get this type of solution by using the generalized symmetry group of the KP equation is worthy of further study. As an explanation, we can write the KP equation (2) in a generalized equivalent form

$$u_t = 6uu_x - u_{xxx} - 3D_x^{-1}u_{yy} + f(y, t) \tag{52}$$

where $f(y, t)$ is an arbitrary function of y and t. It is obvious that an arbitrary function of two variables may be written as a function with infinitely many arbitrary functions of one variable, say,

$$f(y, t) = \sum_{n=-\infty}^{+\infty} A_n(t)y^n \tag{53}$$

where $A_n(t)$, $n=0, \pm 1, \pm 2, \ldots$ are arbitrary functions of t.

In [6] the complete Virasoro algebra constituted by the time-dependent symmetries for various 1 + 1-dimentional integrable models has been obtained. However, the Virasoro algebra of the KP equation we obtained is incomplete because equation (22) holds only for $n, m=0, 1, 2, \ldots$. How to find another 'half' hierarchy for $\tau_n(n=-1, -2, \ldots)$, or even whether there exists such a negative hierarchy is still unknown. Various other interesting problems about the symmetries and the symmetry algebra of the KP equation, like the conservation laws, are also worthy of further investigation. Similar to the KP equation, there may exist generalized symmetries and infinite-dimensional Lie algebras for other 2 + 1-dimensional integrable models. For instance, the generalized symmetries and infinite-dimensional Lie algebras of the integrable dispersive long wave equations in two spaces have been reported in [17].

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